Solution 7

1. Consider maps from $\mathbb{R}$ to itself. Provide explicit examples of continuous maps with exactly one, two and three fixed points.

   Solution. Let f be our function. We consider $g(x) = f(x) - x$. It suffices to produce examples with exactly one, two and three roots. For instance, $g_1(x) = -x$ has exactly one root. $g_2(x) = x^2 - 1$ has exactly two roots. $g_3(x) = (x-1)(x-2)(x-3)$ has exactly three roots. The corresponding f_1, f_2, f_3 fulfil our requirement.

2. Show that the equation $x = \frac{1}{2}\cos^2 x$ has a unique solution in $\mathbb{R}$.

   Solution. Let $Tx = \frac{1}{2}\cos^2 x$. Then $T'(x) = -\frac{1}{2}\sin 2x$ so $|T'| \le 1/2$. It follows that $|Tx - Ty| \le \frac{1}{2}|x-y|$, T is a contraction. By the fixed point theorem, we conclude that $x = \frac{1}{2}\cos^2 x$ has a unique solution.

3. Let T be a continuous map on the complete metric space X. Suppose that for some k, T^k becomes a contraction. Show that T admits a unique fixed point. This generalizes the contraction mapping principle in the case $k = 1$.

   Solution. Since T^k is a contraction, there is a unique fixed point $x \in X$ such that $T^k x = x$. Then $T^{k+1}x = T^k T x = Tx$ shows that Tx is also a fixed point of T^k. From the uniqueness of fixed point we conclude $Tx = x$, that is, x is a fixed point for T. Uniqueness is clear since any fixed point of T is also a fixed point of T^k.

4. Show that the equation $2x\sin x - x^4 + x = 0.001$ has a root near $x = 0$.

   Solution. Here $\Psi(x) = 2x\sin x - x^4$. We need to find some r, γ so it is a contraction. We have

$$\begin{aligned} |\Psi(x_1) - \Psi(x_2)| &= \left|2x_1(\sin x_1 - \sin x_2) + 2(x_1 - x_2)\sin x_2 - (x_1^4 - x_2^4)\right| \\ &= \left|2x_1 \cos c(x_1 - x_2) + 2(x_1 - x_2)\sin x_2 - (x_1^2 + x_2^2)(x_1 + x_2)(x_1 - x_2)\right| \\ &\le (2r + r + (2r^2)(2r))|x_1 - x_2| \ . \end{aligned}$$

   Taking $r = 1/4, \gamma = 2r + r + (2r^2)(2r) = 13/16 < 1$. By the Perturbation of Identity Theorem, the equation $2x\sin x - x^4 + x = y$ is solvable for any y satisfying $|y| \le R = (1-\gamma)r = 0.0468$, including $y = 0.001$.

5. Can you solve the system of equations

$$x + y^4 = 0, \quad y - x^2 = 0.015 \ ?$$

   Solution. Here we work on $\mathbb{R}^2$ and $\Phi(x, y) = (x, y) + \Psi(x, y)$ where $\Psi(x, y) = (-y^4, x^2)$. We have $\Phi(0, 0) = (0, 0)$ and want to solve $\Phi(x_1, x_2) = (0, 0.015)$. In the following points in $\mathbb{R}^2$ are denoted by $p = (x_1, y_1), q = (x_2, y_2)$, etc.

$$\begin{aligned} \|\Psi(p) - \Psi(q)\|_2 &= \|(-y_1^4 + y_2^4, x_1^2 - x_2^2)\|_2 \\ &= \|((y_1^2 + y_2^2)(y_1 + y_2)(y_2 - y_1), (x_1 + x_2)(x_1 - x_2)\|_2 \\ &\le \sqrt{(2r^2 \times 2r)^2 + (2r)^2}\|p - q\|_2 \\ &= 2r(1 + 4r^2)\|p - q\|_2 \ . \end{aligned}$$

   (We have used $|x_1 - x_2|, |y_1 - y_2| \le \|p - q\|_2$.) Hence by taking $r = 1/4, \gamma = 5/8$ and $R = 3/24 = 0.125$. As $0.015 < 0.125$, the system is solvable.

6. Can you solve the system of equations

$$x + y - x^2 = 0, \quad x - y + xy\sin y = -0.002 \ ?$$

Hint: Put the system in the form $x + \cdots = 0, \quad y + \cdots = 0$, first.

Solution. First we rewrite the system in the form of $I + \Psi$. Indeed, by adding up and subtracting the equations, we see that the system is equivalent to

$$x + (-x^2 + xy\sin y)/2 = -0.001, \quad y + (-x^2 - xy\sin y)/2 = 0.001 \ .$$

Now we can take

$$\Psi(x, y) = \frac{1}{2}(-x^2 + xy\sin y, -x^2 - xy\sin y) \ ,$$

and proceed as in the previous problem.

7. Let $A = \{a_{ij}\}$ be an $n \times n$ matrix. Show that

$$|Ax| \le \sqrt{\sum_{i,j} a_{ij}^2} \ |x|.$$

Solution. Let $y = Ax$. We have

$$y_i = \sum_j a_{ij}x_j, \quad i = 1, \cdots, n \ .$$

By Cauchy-Schwarz Inequality,

$$|y_i| \le \sqrt{\sum_j a_{ij}^2}\sqrt{\sum_j x_j^2} \ .$$

Taking square,

$$y_i^2 \le \sum_j a_{ij}^2 \sum_j x_j^2 \ .$$

Summing over i,

$$\sum_i y_i^2 \le \sum_{i,j} a_{ij}^2 \sum_j x_j^2 \ ,$$

and the result follows by taking root.

Note. This result was used in the proof of Proposition 3.5.

8. Let $A = (a_{ij})$ be an $n \times n$ matrix. Show that the matrix $I + A$ is invertible if $\sum_{i,j} a_{ij}^2 < 1$. Give an example showing that $I + A$ could become singular when $\sum_{i,j} a_{ij}^2 = 1$.

Solution. Let $\Phi(x) = Ix + Ax$ so that $\Psi(x) = Ax$ for $x \in \mathbb{R}^n$. By the previous problem,

$$|\Psi(x_1) - \Psi(x_2)| = |A(x_1 - x_2)| \le \sqrt{\sum_{i,j} a_{ij}^2} \ |x| \ .$$

Take $\gamma = \sqrt{\sum_{i,j} a_{ij}^2} < 1$. Ψ is a contraction and there is only one root of the equation $\Phi(x) = 0$ in the ball $B_r(0)$. However, since we already know $\Phi(0) = 0$, 0 is the unique root. Now, we claim that $I + A$ is non-singular, for there is some $z \in \mathbb{R}^n$ satisfying $(I + A)z = 0$, we can find a small number α such that $\alpha z \in B_r(0)$. By what we have just shown, $\alpha z = 0$ so $z = 0$, that is, $I + A$ is non-singular and thus invertible.

The sharpness of the condition $\sum a_{ij}^2 < 1$ can be seen from considering the 2×2-matrix A where all $a_{ij} = 0$ except $a_{22} = -1$.

Note. See how linearity plays its role in the proof.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0) = 0, f'(x_0) \neq 0$. Show that there exists some $\rho > 0$ such that
$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho),$$
is a contraction. This provides a justification for Newton's method in finding roots for an equation.

Solution. $T'(x) = \dfrac{f(x)f''(x)}{f'(x)^2}$. Since f is C^2 and $f(x_0) = 0, f'(x_0) \neq 0$, it follows that T is C^1 in a neighborhood of x_0 with $T(x_0) = x_0, T'(x_0) = 0$ and there exists some $\rho > 0$
$$|T'(x)| \leq \frac{1}{2}, \quad x \in [x_0 - \rho, x_0 + \rho].$$
As a result, T is a contraction in $[x_0 - \rho, x_0 + \rho]$. By Contraction Mapping Principle, there is a fixed point for T. From the definition of T, this fixed point is a root for the equation $f(x) = 0$.

10. Consider the iteration
$$x_{n+1} = \alpha x_n(1 - x_n), \ x_1 \in [0, 1] \ .$$
Find

(a) The range of α so that $\{x_n\}$ remains in $[0, 1]$.

(b) The range of α so that the iteration has a unique fixed point 0 in $[0, 1]$.

(c) Show that for $\alpha \in [0, 1]$ the fixed point 0 is attracting in the sense: $x_n \to 0$ whenever $x_0 \in [0, 1]$.

Solution. Let $Tx = \alpha x(1 - x)$. The max of T attains at $1/2$ so the maximal value is $\alpha/4$. Therefore, the range of α is $[0, 4]$ so that T maps $[0, 1]$ to itself. Next, 0 is always a fixed point of T. To get no other, we set $x = \alpha x(1 - x)$ and solve for x and get $x = (\alpha - 1)/\alpha$. So there is no other fixed point if $\alpha \in [0, 1]$. Finally, it is clear that T becomes a contraction when $\alpha \in [0, 1)$, so the sequence $\{x_n\}$ with $x_0 \in [0, 1]$, $x_n = T^n x_0$, always tends to 0 as $n \to \infty$. Although T is not a contraction when $\alpha = 1$, one can still use elementary mean (that is, $\{x_n\}$ is always decreasing,) to show that 0 is an attracting fixed point.